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Symmetry distribution between hook length and part length for partitions

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ABSTRACT

It is known that the two statistics on integer partitions "hook length" and "part length" are equidistributed over the set of all partitions of n. We extend this result by proving that the bivariate joint generating function by those two statistics is symmetric. Our method is based on a generating function by a triple statistic much easier to calculate.

1. Introduction

The basic notions needed here can be found in [11, p. 287]. A *partition* λ is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. The integers λ_i, $i = 1, 2, \ldots, \ell$ are called the *parts* of λ, the number ℓ of parts being the *length* of λ denoted by $\ell(\lambda)$. The sum of its parts $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ is denoted by $|\lambda|$. Let n be an integer; a partition λ is said to be a partition of n if $|\lambda| = n$. We write $\lambda \vdash n$.

Each partition can be represented by its Ferrers diagram (or Young diagram). For each box v in the Ferrers diagram of a partition λ, or for each box v in λ, for short, define the *arm length* (resp. *leg length*, *coarm length*, *coleg length*) of v, denoted by a_v or $a_v(\lambda)$ (resp. l_v, m_v, g_v), to be the number of boxes u such that u lies in the same row as v and to the right of v (resp. in the same column as v and above v, in the same row as v and to the left of v, in the same column as v and under v). See Fig. 1.

We define the *hook length* (resp. *part length*) of v in λ to be $h_v = a_v + l_v + 1$ (resp. $p_v = m_v + a_v + 1$). Bessenrodt [3], Bacher and Manivel [2] have proved that the two statistics h_v and p_v are equidistributed over the set of all partitions of n, i.e.,

$$\sum_{\lambda \vdash n} \sum_{v \in \lambda} x^{h_v} = \sum_{\lambda \vdash n} \sum_{v \in \lambda} x^{p_v}. \tag{1}$$

For example, the set of all partitions of 4 with their hook lengths (resp. part lengths) is reproduced in Fig. 2 (resp. Fig. 3). We see that the above two generating functions by h_v and by p_v are identical $7x + 6x^2 + 3x^3 + 4x^4$.

Previous studies have been done along those lines by Stanley, Elder, Schmidt and Simion, Hoare, Kirdar, Skyrme, Han and Ji [10,8,12,9,13,14,5–7]. In particular, it was shown that the product over all parts of all partitions of a fixed number n equals

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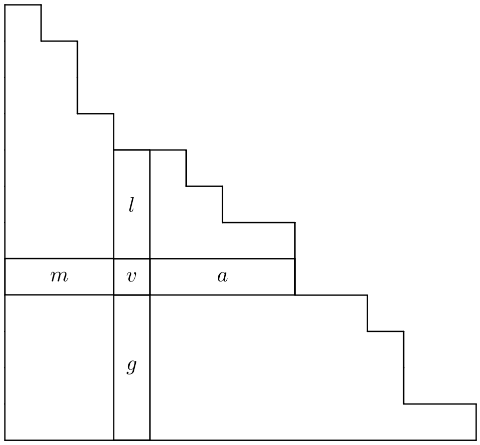

Fig. 1. Arm, leg, coarm, coleg lengths.

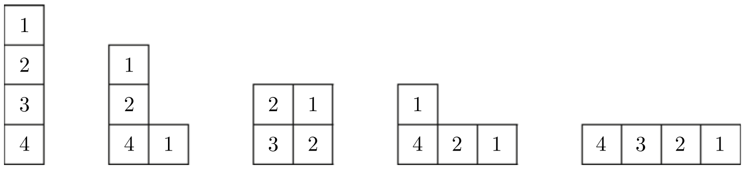

Fig. 2. Partitions of 4 and the hook lengths h_v.

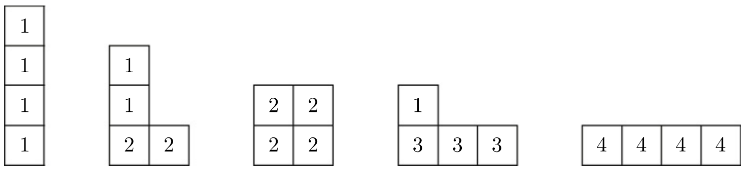

Fig. 3. Partitions of 4 and the part lengths p_v.

the product over the factorials of all part multiplicities in all partitions of n. The combinatorial proofs of this identity give in fact that the multisets of the corresponding factors in the products are equal. This may be interpreted as saying that for all k, the number of parts k in all partitions of n equals the number of k-hooks of arm length 0 in all of these partitions.

In the present paper we study the joint distribution of the two statistics hook length h_v and part length p_v. Our main result is the following theorem.

Theorem 1. *The bivariate joint generating function for the partitions of n by the two statistics h_v and p_v is symmetric. In other words, let*

$$P_n(x,y) = \sum_{\lambda \vdash n} \sum_{v \in \lambda} x^{h_v} y^{p_v}. \tag{2}$$

We have

$$P_n(x,y) = P_n(y,x).$$

For example, the joint distribution of h_v and p_v for the partitions of 4 is reproduced in the following tableau, which is symmetric.

$p \setminus h$	1	2	3	4	$\sum$
1	3	2	1	1	7
2	2	2	1	1	6
3	1	1	0	1	3
4	1	1	1	1	4
$\sum$	7	6	3	4	20

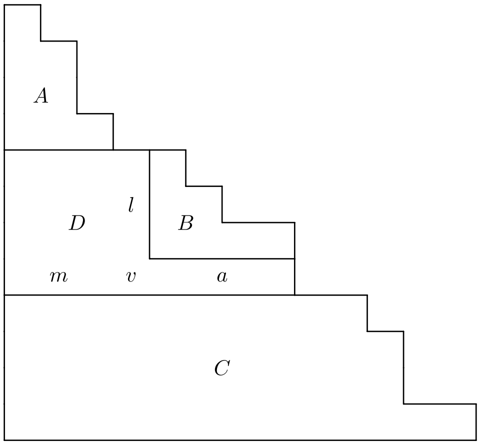

Fig. 4. Partition and its regions.

2. The proof

First, recall the usual notation of the q-ascending factorial [4, chap. 1]

$$(a;q)_n = \begin{cases} 1, & \text{if } n = 0; \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n \geq 1. \end{cases}$$

For $0 \leq k \leq n$ let $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}$ be the usual q-binomial coefficient.

We prove the following more precise result which will easily lead to a proof of Theorem 1. For each given triplet (a, l, m) of integers let $f_n(a, l, m)$ denote the number of the ordered pairs (λ, v) such that $\lambda \vdash n$, $v \in \lambda$, $a_v = a$, $l_v = l$, $m_v = m$. We obtain the explicit generating function for $f_n(a, l, m)$.

Theorem 2. *The generating function of $f_n(a, l, m)$ is given by the following formula:*

$$\sum_{n\geq 0} f_n(a,l,m)q^n = \frac{(q;q)_a}{(q;q)_\infty} \begin{bmatrix} l+a \\ a \end{bmatrix}_q \begin{bmatrix} m+a \\ a \end{bmatrix}_q q^{(m+1)(l+1)+a}.$$

Proof. For a fixed partition $\lambda \vdash n$ it is easy to see that all triplets (a_v, l_v, m_v) (for $v \in \lambda$) are different. Now, let the triplet (a, l, m) be fixed and the partition λ be free; the number of pairs (λ, v) such that $v \in \lambda$, $a_v = a$, $l_v = l$, $m_v = m$ is equal to the number of partitions λ, such that there is a box $v \in \lambda$ with $a_v = a$, $l_v = l$, $m_v = m$. The generating function for those partitions is equal to the product of several "small" generating functions for the regions of the partitions, as shown in Fig. 4.

Let $F(a, l, m; q) = \sum_n f_n(a, l, m)q^n$. It is quite routine (see, e.g., [1, chap. 3]) to prove that

$$F(a,l,m;q) = A(q)B(q)C(q)D(q),$$

where

$$A(q) = 1/(q;q)_m;$$
$$B(q) = \begin{bmatrix} l+a \\ a \end{bmatrix}_q;$$
$$C(q) = \frac{1}{(1-q^{m+a+1})(1-q^{m+a+2})\cdots} = \frac{(q;q)_{m+a}}{(q;q)_\infty};$$
$$D(q) = q^{(m+1)(l+1)+a}.$$

Finally, we obtain the generating function $F(a, l, m; q)$ by multiplying all the four expressions given above. $\square$

Theorem 3. *. The triple statistic (a_v, l_v, m_v) has the same distribution as (a_v, m_v, l_v). In other words, let the generating function for (a_v, l_v, m_v) be*

$$Q_n(x,y,z) = \sum_{\lambda\vdash n}\sum_{v\in\lambda} x^{a_v}y^{l_v}z^{m_v}.$$

Then

$$Q_n(x,y,z) = Q_n(x,z,y).$$

Proof. It suffices to prove the symmetry property for all the coefficients in Q_n. For each triple of integers (a, l, m) we have to show that

$$[x^a y^l z^m]Q_n(x, y, z) = [x^a y^m z^l]Q_n(x, y, z)$$

or $f_n(a, l, m) = f_n(a, m, l)$, which is true by Theorem 2. □

Proof of Theorem 1. By Theorem 3 we have

$$P_n(x, y) = xyQ_n(xy, x, y) = xyQ_n(xy, y, x) = P_n(y, x).\quad \square$$

3. Super-symmetry

Let $U(x, y)$ be a polynomial in x and y. We say that U is *super-symmetric* on x and y, if $[x^\alpha y^\beta]U(x, y) = [x^{\alpha'} y^{\beta'}]U(x, y)$ when $\alpha + \beta = \alpha' + \beta'$. In particular, any super-symmetric polynomial is also symmetric. Bessenrodt [3], Bacher and Manivel [2] have obtained the following hook-type theorem, which is more general than the equidistribution property (see (1)). It can also be proved directly using our result.

Theorem 4. *. The bivariate joint generating function for the partitions of n by the two joint statistics a_v and l_v is super-symmetric. In other words, let*

$$G_n(x, y) = \sum_{\lambda \vdash n} \sum_{v \in \lambda} x^{a_v} y^{l_v}.$$

Then $[x^\alpha y^\beta]G(x, y) = [x^{\alpha'} y^{\beta'}]G(x, y)$ *when* $\alpha + \beta = \alpha' + \beta'$.

Proof. Let $\alpha + \beta = \alpha' + \beta'$. Let λ be a partition, $v \in \lambda$ with $(a_v, m_v, g_v) = (\alpha, \beta, g)$. Then, there is a unique box $u \in \lambda$ satisfying $(a_u, m_u, g_u) = (\alpha', \beta', g)$. Hence, the bivariate joint generating function for the partitions of n by the two statistics a_v and m_v

$$\sum_{\lambda \vdash n} \sum_{v \in \lambda} x^{a_v} y^{m_v}$$

is super-symmetric. By Theorem 3, $G_n(x, y)$ is also super-symmetric. □

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